

# Elements of a new approach to time in Quantum Mechanics

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In this work we present a re-evaluation of the concept of time in non-relativistic quantum theory. We suggest a formalism in which time is changed into the status of an operator, and where expectation values of observables and the state of a quantum system are reworked. This approach leads us to an additional concept, given by a temporal probability distribution associated with the actual measurement of an observable.

## I. INTRODUCTION

In Schrödinger quantum mechanics there is a clear asymmetry between time and space. Time is treated as a continuous parameter that can be chosen with arbitrary precision and employed to label the solution of the wave equation. A character of infinite divisibility of time is taken for granted. In contrast, the position of a particle is treated as an operator, and therefore, the knowledge of its value becomes inherently probabilistic. It is quite common to find the reasoning that this asymmetry is due to the non-relativistic character of the Schrödinger equation. Although partially correct, this argument is insufficient to justify all the disparity between space and time in the formalism of quantum mechanics. An illustration is as follows: the squared modulus of the wave function provides the probability, in a position measurement, of finding the particle between  $x$  and  $x + dx$  exactly at the instant  $t$ . Would it not be more reasonable and symmetrical, even in the non-relativistic domain, to ask about the probability of a particle to be measured between  $x$  and  $x + dx$  at an instant between  $t$  and  $t + dt$ ?

The first goal of this manuscript is to extend certain concepts in quantum mechanics and show that the asymmetry between  $t$  and  $x$  is, in fact, not entirely due to the lack of Lorentz covariance of the theory. Firstly, the study proposed here is motivated by the uncertainty relation between energy and time. The inequality  $\Delta E \Delta T \geq \hbar/2$  has more diverse interpretations than those involving two canonically conjugate observables. Consider the case of a stationary state, for which we can determine with certainty the energy of the system and, hence,  $\Delta E = 0$ . Consequently, we should have “ $\Delta T = \infty$ ”, so that a plausible interpretation of  $\Delta T$  would be the lifetime of the state. Alternatively, in his book “Quantum Mechanics” [1], D. J. Griffiths presents a derivation of the energy-time uncertainty relation, in which, an observable  $\hat{A}$  is measured and  $\Delta T$  plays the role of the time required for  $\langle A \rangle$  to vary by a standard deviation of  $\hat{A}$ . Therefore, by D. J. Griffiths “ $\Delta T = \infty$ ” indicates that the expectation value of the associated observable is constant in time. If the operator  $\hat{A}$  and the

Hamiltonian do not commute, these two interpretations not necessarily coincide.

It is possible to think of quantum dynamics as a combination of two quite disparate processes: a unitary evolution, and the collapse caused by measurements. Given a  $|\psi(t)\rangle$  satisfying the first process, in quantum theory we can select  $t$  with arbitrary precision to analyse possible results of the second process. The fact is that this arbitrary temporal choice does not take into account the fundamental aspect that any information about a quantum system is only obtained, in practice, by a measurement that has an intrinsic temporal randomness. A second goal of this work is to review the concepts of quantum mechanics associated with this analysis, taking into consideration the fact that we do not have an infinitely precise knowledge of the time in which we “observe” the system. In order to do this, we abandon the idea that one can describe the state of a system at a given instant of time. In this case, we can also define a time operator that provides a starting point for a more systematic approach to the concept of time in non-relativistic quantum mechanics.

Operators related to time have already been addressed in several different contexts [3–12]. Among these approaches, an important problem is related to the arrival time of a particle in an apparatus that is spatially localized. In this scenario, a time operator is defined so that the relation  $[\hat{T}, \hat{H}] = i\hbar$  is satisfied, and the main objective is to obtain the probability distribution for the time of arrival at the detector, quite a useful concept, especially from an experimental point of view. This idea brought significant academic interest and numerous case studies, as for example in quantum tunneling [8]. We will see later that the physical motivation and the way the operator associated with the arrival time is defined is quite different and, indeed, complementary with respect to the one proposed in this work. Some of our results on this topic are discussed in next section.

We should point out that a very recent work entitled “Quantum time” [12] proposed a new, consistent way to understand time in quantum mechanics. Although we followed a distinct path, our equation (10) is equivalent to Eq. (23) of Ref. [12]. The study carried out in Ref. [12] is based in a concept, originally proposed by Page and Wootters [13], of the conditional character of the quantum state relative to a Hamiltonian system defined as a clock. Our approach embraces the same conditional in-

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interpretation for the solution of the Schrödinger equation by using quite different arguments and physical motivation.

## II. SPACE-TIME SYMMETRY AND THE ROLE OF TIME IN QUANTUM MECHANICS

Some symmetries between time and position, often concealed by the standard presentation of the theory, can be found in operators and equations of non-relativistic quantum mechanics. For instance:

$$\hat{H}\hat{U}_t(t, t') = i\hbar \frac{d}{dt} \hat{U}_t(t, t') \quad (1)$$

and

$$\hat{P}\hat{U}_x(x, x') = i\hbar \frac{d}{dx} \hat{U}_x(x, x'), \quad (2)$$

where  $\hat{H}$  is the Hamiltonian of the system,  $\hat{U}_t$  is the time evolution operator (or temporal translation operator),  $\hat{P}$  is the momentum operator, and  $\hat{U}_x$  is the spatial translation operator. By acting the ket  $|\psi(t')\rangle$  (solution of the Schrödinger equation) on the right hand side of Eq. (1), and acting  $|x'\rangle$  in the same way in Eq. (2), we obtain

$$\hat{H}|\psi(t)\rangle = i\hbar \frac{d}{dt} |\psi(t)\rangle \quad (3)$$

and

$$\hat{P}|x\rangle = i\hbar \frac{d}{dx} |x\rangle, \quad (4)$$

where the interplay between pairs  $\hat{H}$  and  $\hat{P}$ ,  $\hat{U}_t$  and  $\hat{U}_x$ , and *also* between  $|\psi(t)\rangle$  and  $|x\rangle$  is clear. Because of the similarity between Eqs. (3) and (4), we are compelled to tackle the delicate problem of defining a time operator  $\hat{T}$  whose eigenvalues correspond to the very solutions of the time-dependent Schrödinger equation for each instant of time. So, defining  $|\psi(t)\rangle \equiv |t\rangle$ , where  $\hat{T}|t\rangle = t|t\rangle$ , we have the following relations

$$\hat{H}|t\rangle = i\hbar \frac{d}{dt} |t\rangle, \quad \hat{P}|x\rangle = i\hbar \frac{d}{dx} |x\rangle, \quad (5)$$

and

$$\hat{T}|t\rangle = t|t\rangle, \quad \hat{X}|x\rangle = x|x\rangle. \quad (6)$$

Note that  $x$  and  $t$  are placed in a quite symmetrical framework. However, one can easily note that  $\hat{T}$  is non-Hermitian, whereas  $\hat{X} = \hat{X}^\dagger$ . This has been one of the biggest concerns regarding the definition of a time operator.

It is worth to point out that a similar operator was defined in Ref. [11] through a distinct physical reasoning. In [11] a picture of quantum mechanics is assumed in which the representation vectors evolve in time and

state vectors and operators are static, which lead to very different conclusions in comparison to ours.

The approach described above seems to provide interesting physical consequences, while it does not affect the existing results in quantum mechanics. Consider, for example, a Hamiltonian with an explicit time dependence. According to our suggestion, we should replace the parameter  $t$  by the operator  $\hat{T}$ . However, in the Schrödinger equation, as the Hamiltonian acts in the eigenvector of  $\hat{T}$ ,  $|t\rangle$ , time becomes a scalar quantity again, and, thus, nothing is changed in comparison with standard quantum theory. Another important consequence of the existence of this operator is that we can easily obtain the relation  $[\hat{T}, \hat{H}] = i\hbar$ . However, by looking at the dispersion of  $\hat{T}$  a problem arises, which leads us to rework some concepts as follows.

The calculation of the expectation value of the time operator for a physical system whose ket state is  $|\psi(t)\rangle$  reads

$$\langle\psi(t)|\hat{T}|\psi(t)\rangle = \langle t|\hat{T}|t\rangle = t \langle t|t\rangle = t, \quad (7)$$

which implies that the temporal uncertainty vanishes,  $\Delta T = \langle T^2 \rangle - \langle T \rangle^2 = 0$ , regardless of the details of the physical system considered. As discussed earlier, the reason why  $\Delta T = 0$  is the usual approach of quantum mechanics that considers time as a parameter that can be chosen arbitrarily. Moreover, this result, conceptually unsatisfactory, could not take us to the Heisenberg inequality via first principles. Therefore, we are lead to a redefinition of quantum expectation values. Let  $\hat{A}$  be an observable, the modified expectation value of  $\hat{A}$  reads

$$\langle\langle\hat{A}\rangle\rangle \equiv \int_0^\infty f(t) \langle\hat{A}\rangle dt. \quad (8)$$

Eq. (8) corresponds to a time average of the usual expectation value of quantum mechanics  $\langle\hat{A}\rangle$  weighted by the function  $f(t)$ , which satisfies the condition  $\int_0^\infty f(t) dt = 1$ . The function  $f(t)$  is interpreted as the temporal probability distribution for the wave function to collapse once the system is under measurement. We stress that the formalism described from this point on is intended to be valid only in the temporal window where the collapse can potentially occur, that is, after the arrival time mentioned in the introduction has elapsed. A similar interpretation appears in a distinct, more specific context, in studies on physical collapse dynamics [14, 15].

Note, in particular, that for  $\hat{A} = \hat{T}$ , we have

$$\langle\langle\hat{T}\rangle\rangle = \int_0^\infty f(t) t dt, \quad (9)$$

where a clear difference from the expectation value given by Eq. (7) is observed. By using the interpretation of  $f(t)$  given earlier, Eq. (9) yields the average time for the collapse to occur in a set of measurements over an ensemble of equally prepared physical systems. Here a key concern arises: what kind of quantum state would be

consistent with the expression of Eq. (8)? To answer this question, we redefine the quantum state of the system using the density operator formalism. The quantum state of a system *under measurement*, with the solution of the Schrödinger equation given by  $|t\rangle$ , is defined as

$$\hat{\Omega} = \int_0^\infty f(t)|t\rangle\langle t|dt. \quad (10)$$

Note that the state of Eq. (10) does not correspond to a representation of the usual density operator of quantum mechanics obtained by using the states  $|\psi(t)\rangle = |t\rangle$ . In fact, these states are not even a basis for the Hilbert space. Eq. (10) represents a new quantum state given by the sum of the Schrödinger equation solutions for each instant of time, weighted by the function  $f(t)$ . Note that Eq. (10) does not consider the time as a parameter whose value can be arbitrarily chosen. One can recover the standard theory by setting  $f(t) = \delta(t - t')$ . As mentioned in the introduction, one can obtain the state (10) through the state defined in Eq. (23) of Ref. [12]. In order to do this, one traces out the clock space, defined by the authors as an ancillary system.

Note that the quantum state proposed in this work corresponds to a temporal convolution of solutions to the Schrödinger equation, being independent of space and time. This approach is markedly distinct from that of standard quantum mechanics, where the state is described for a particular time. Furthermore, every physical theory should describe the results of observations which, in some instance, correspond to classical concepts that make our comprehension viable. The suggestions presented here produce probabilities that an observation will be materialised in a given time window and in a given region of space. In this way, to infer about the state of a particle at a given time  $t$ , makes as much sense as to ask about the state of that particle in a given position  $x$ . Therefore, the physical description of a quantum state in a fixed time that instantly changes due to a measurement is incompatible with the description presented here. By the same token, consider the probabilistic process of spontaneous decay of an atom. It is described by a superposition  $(|e\rangle|0\rangle + |g\rangle|1\rangle)/\sqrt{2}$  that contemplates the existence and the non-existence of a photon. Whether or not the photon exists immediately before the measurement (whether or not the cat is dead or alive) becomes an ill-posed question in the presented framework.

With these ideas in mind, consider a Stern-Gerlach experiment in which one of the detectors is removed. If no click is observed after a certain period of time, the collapse of the the quantum system occurs so that one can conclude that the atom ran through the path where the detector was removed. In these circumstances, one can ask about either what is the state of the particle in a given instant of time or, in other words, what is the moment at which the Schrödinger evolution is no longer valid and the instantaneous change of the state takes place. However, notice that since (10) is a temporal superposition of Schrödinger solutions, the very validity of this kind of

question becomes disputable.

According to one of the mantras of the Copenhagen interpretation, the idea that the position of particle becomes a physical reality only when a measurement is made. It is a tenable position to expect that the concept of time should emerge in the same way. Successive observations would give us the notions of space and time that we classically experience.

It is important to note that all usual properties of the density operator are kept for the state (10), for example

$$\text{Tr } \hat{\Omega} = \mathbb{I}, \quad (11)$$

and

$$\text{Tr}(\hat{\Omega}\hat{A}) = \langle\langle\hat{A}\rangle\rangle = \int_0^\infty f(t)\langle\hat{A}\rangle dt, \quad (12)$$

where  $\mathbb{I}$  is the identity operator. In this formalism, the probability of finding a particle between  $x$  and  $x + dx$  at an infinitely precise instant of time  $t$  must be zero. However, the probability of finding the particle between  $x$  and  $x + dx$  in the the interval of time  $t$  and  $t + dt$  is given by

$$\begin{aligned} p(x, t) dx dt &= f(t)|\langle x|t\rangle|^2 dx dt \\ &= f(t)|\psi(x, t)|^2 dx dt. \end{aligned} \quad (13)$$

This new definition of quantum probability can be obtained from Eq. (10) by calculating

$$\text{Tr}(\hat{\Omega}|a\rangle\langle a|) = \int_0^\infty f(t)|\langle a|t\rangle|^2 dt, \quad (14)$$

and by assuming  $|a\rangle = |x\rangle$ .

By inspecting Eq. 13, we realize the conditional character of the probability density  $|\psi(x, t)|^2$ . According to the Bayes rule, one can assert that the probability of finding the particle between  $x$  and  $x + dx$  in the the interval of time  $t$  and  $t + dt$  is equal to the probability of finding the particle between  $x$  and  $x + dx$  *given that* the measurement occurred in the interval  $[t, t + dt]$  times the probability for the system to be measured between the instants  $t$  and  $t + dt$ :

$$\begin{aligned} p(x, t) dx dt &= P(x|t)P(t) dx dt \\ &= |\psi(x, t)|^2 f(t) dx dt. \end{aligned} \quad (15)$$

In other context, a conditional property of the wave function has already been addressed in Ref. [13] and reinforced in Ref. [12]. Notice that  $|\psi(x, t)|^2$  is interpreted now as the probability density of finding the particle at the position  $x$  given that the measurement occurred at the time  $t$ , so that perhaps we should express the wave function with a more provocative notation, such as  $\psi(x|t)$ . It is essential to note that by strictly following Bayes rule, one must not assume that the function  $f(t)$  can be obtained by the knowledge of  $|\psi(t)\rangle = |t\rangle$ , the solution of the Schrödinger equation. It would correspond to essentially new information necessary to express the

full state of the system as in Eq. (10). In addition, by the symmetry of Bayes rule, in principle, we can express the probability of finding a particle in  $[x, x + dx]$  and  $[t, t + dt]$  as

$$\begin{aligned} p(x, t) \, dx dt &= P(t|x)P(x) \, dx dt \\ &= |\phi(t|x)|^2 g(x) \, dx dt, \end{aligned} \quad (16)$$

where  $|\phi(t|x)|^2$  corresponds to the probability density of finding the particle at the instant  $t$  given that the result of the position measurement is  $x$ , and  $g(x)$  is the density probability for the system to be measured at  $x$ .

We stress that the average time given by Eq. (9) has a different meaning from that of the expectation value of the “arrival time” operator, discussed earlier. The calculation of Eq. (9) provides an average time for the quantum state to collapse which, in general, is much shorter than the average time of arrival of a particle in an actual experimental situation. In our case, we treat the system as if it had already “arrived” at the detector and we do not know at what time the collapse should occur.

### III. PERSPECTIVES AND CONCLUSIONS

As a perspective, a first step would be to obtain constraints on  $f(t)$  or even to propose an analytical form for this distribution, and to test its predictions in measurement processes. Furthermore, we lack a deeper under-

standing of the unconventional aspects, which naturally arise, such as the non-Hermiticity of the time operator. Several other fundamental questions arise due to these new concepts: is the function  $f(t)$  a property of the quantum system itself or should it be described considering the physical characteristics of the measuring apparatus? As  $f(t)$  refers to the probability of collapse, does this temporal distribution have universal properties which are independent of the observable that is being measured? It would be also interesting to apply this new approach to physical situations widely discussed in the literature, for example, tunneling. Besides, it is important to make a careful analysis of these concepts in comparison to previous studies related to time operators [3–10]. We believe that further developments in the presented model and its experimental investigation may help us to get a better understanding of the mechanism of quantum collapse [14, 15] and of the role of time in quantum theory. Comments and suggestions on these preliminary ideas are very welcome.

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